Considerations on Hydrodynamics

by Eugenio Beltrami

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In the general theory of the motion of fluids, there are two doubly infinite systems of lines that have a fundamental importance for kinematic and dynamic studies of the motion itself. One of these systems is that of the lines of flux, defined by the differential equation

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}, \quad dt = 0,$$

where u, v, w, are the velocity components at the point (x, y, z) and at the instant t; the other system is the vortical lines, defined by the differential equation

$$\frac{dx}{p} = \frac{dy}{q} = \frac{dz}{r}, \quad dt = 0,$$

where p, q, r are the components of the rotation at the same point, and are defined by the well-known equation

$$2p = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad 2q = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x},$$

$$2r = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

These two systems of lines are not, evidently, independent of each other, even if their mutual dependence is not explicit. They appear only in a very indirect way in the well-known hydrodynamic theorems. We will not speak about this question in general, but only about two cases that can be considered as extreme cases. The first is the case in which the lines of the two systems cross each other at right angles in each instant of time and in each point of the space occupied by the fluid. It is defined by the equation

$$pu + qv + rw = 0$$

which must be true for the whole duration of the motion, throughout the whole space. This equation has a well-known interpretation, which expresses the necessary and sufficient condition in which the trinomial

$$udx + vdy + wdz$$

always can be integrated. The class of motion of the fluid in which this property is true is fully represented by the formulae

$$u = \mu\frac{\partial\varphi}{\partial x}, \quad v = \mu\frac{\partial\varphi}{\partial y}, \quad w = \mu\frac{\partial\varphi}{\partial z},$$

where μ and ϕ are two arbitrary functions of the space and time coordinates.

On the contrary, the second case takes place when these lines always meet at a 0 angle. In other words, they coincide at each point in time and space. The analytic conditions for this coincidence are

$$\frac{p}{u} = \frac{q}{v} = \frac{r}{w}, \tag{1}$$

otherwise

$$qw - rv = 0, \; ru - pw = 0, \; pv - qu = 0 \tag{1$_a$}$$

which of these last equations, it can be said that one of them can be derived from the other two. Can this second case actually be verified? If not, then, with the hypotheses,

$$u = \frac{\partial \varphi}{\partial x}, \quad v = \frac{\partial \varphi}{\partial y}, \quad w = \frac{\partial \varphi}{\partial z},$$

p, q, r must be 0, which means that the vortical lines do not exist.

We shall begin by considering a particular class of motions, in which each molecule of fluid moves in parallel to a fixed plane, which we suppose to be the xy plane. In this case we have $w = 0$ and then

$$2p = -\frac{\partial v}{\partial z}, \quad 2q = \frac{\partial u}{\partial z}, \quad 2r = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

That means that equations (1_a) are the following:

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0, \quad u\frac{\partial u}{\partial z} + v\frac{\partial v}{\partial z} = 0.$$

Providing that the following conditions hold

$$u = \frac{\partial \varphi}{\partial x}, \quad v = \frac{\partial \varphi}{\partial y}, \quad w = 0, \qquad (2)$$

where ϕ is a function of x, y, z, and t, subject to the condition

$$\frac{\partial}{\partial z}\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2\right] = 0.$$

A particular manner, but sufficient for our purposes, to solve this equation is this. Let F be an arbitrary function of the complex binomial $(x + iy)$ and the time t, and let Z be another arbitrary function, which however has only real values, and of z again take the time as t, posing

$$Fe^{iZ} = \phi + i\psi, \qquad (2_a)$$

which means that the real part is denoted with the symbol ϕ, and the coefficient of the imaginary unity with ψ. The function ϕ satisfies the condition just found. In fact we have

$$2\phi = Fe^{iZ} + F_1 e^{-iX},$$

where F_1 is the conjugate function of F, we have

$$2\frac{\partial \varphi}{\partial x} = F' e^{iZ} + F_1' e^{-iZ},$$

$$2\frac{\partial \varphi}{\partial y} = iF' e^{iZ} - iF_1' e^{-iZ},$$

where the prime indicates finding the derivative with respect to the binomial $(x + iy)$. From which we results

$$\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 = F' F_1',$$

and because the second member (F', F'_1) by hypothesis depends only upon the variables x, y and t, it is clear that the derivative with respect to the variable z is 0, which is the result which we sought.

We have, at least in the case in which motion is parallel to a fixed plane, a class of real motions in which what we require, that is, the coincidence of flux and vortical lines, exists. We note that each function ϕ, which we have created through the previous process, satisfies the equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

which means that this class of motion refers to an incompressible fluid (2). But we note that, having

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x},$$

the differential equations of the flux lines become

$$d\psi = 0, \; dz = 0, \; dt = 0,$$

that means these lines, identical with the vortical lines, are represented by the finite equations

$$\psi = \text{const}, \; z = \text{const}, \; t = \text{const} \qquad (2_b)$$

We can make a very simple example.
Taking

$$F = x + iy, \; Z = -2Tz$$

where T is some function of t, we find

$$Fe^{iZ} = (x + iy)e^{-2iTz},$$

from which

$$\varphi = x \cos 2Tz + y \sin 2tz$$
$$\psi = -x \sin 2Tz + y \cos 2Tz.$$

thus the solution is obtained

$$u = \cos 2Tz,\ v = \sin 2Tz,\ w = 0,$$

in which the required property is immediately verifiable, since we find

$$p = -T\cos 2Tz,\ q = -T\sin 2Tz,\ r = 0$$

and from which

$$\frac{p}{u} = \frac{q}{v} = -T.$$

The flux and vortical lines are straight lines,

$$-x\sin 2Tz + y\cos 2Tz = \text{const},$$
$$z = \text{const},\ t = \text{const}.$$

This particular example easily leads to another example which also relates to an incompressible fluid, but in which the fluid molecules no longer move more parallel in a plane. If, in fact, we take

$$u = T_2\cos 2Ty + T_3\sin 2Tz,$$
$$v = T_3\cos 2Tz + T_1\sin 2Tx,$$
$$w = T_1\cos 2Tx + T_2\sin 2Ty,$$

where T, T_1, T_2, and T_3 are four arbitrary functions of time, we immediately find

$$\frac{p}{u} = \frac{q}{v} = \frac{r}{w} = T.$$

Another class of solutions can be indicated, in which the motion is neither parallel to the plane nor in general relates to an incompressible fluid.

Let ϕ be a general function of x, y, and t, then pose

$$u = -\frac{\partial\varphi}{\partial y}, \qquad v = \frac{\partial\varphi}{\partial x},$$

keeping the third component w, indeterminate for the moment. From this is derived

$$2p = \frac{\partial w}{\partial y},\quad 2q = -\frac{\partial w}{\partial x},\quad 2r = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}.$$

The third equation (1_a) will become

$$\frac{\partial\varphi}{\partial x}\frac{\partial w}{\partial y} - \frac{\partial\varphi}{\partial y}\frac{\partial w}{\partial x} = 0$$

and that shows that w must have the form

$$w = w(\varphi, z, t),$$

from this results

$$\frac{2p}{u} = \frac{2q}{v} = -\frac{\partial w}{\partial\varphi}.$$

The equality of the first two relationships (1) with the third is thus expressed by the equation

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{1}{2}\frac{\partial(w^2)}{\partial\varphi} = 0.$$

But, because ϕ is an independent function of w, and because of this same equation,

$$\frac{\partial^2(w^2)}{\partial\varphi\partial z} = 0,$$

then w^2 must have the form

$$w^2 = F(\varphi, t) + Z(z, t),$$

and ϕ must satisfy the equation

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{1}{2}\frac{\partial F}{\partial\varphi} = 0.$$

Supposing that ϕ and F depend only upon $\rho = \sqrt{x^2+y^2}$ and from t, this equation becomes

$$2\phi'(\rho\phi')' + \rho F' = 0,$$

where the prime indicates the derivative with respect to ρ. In these particular hypotheses, the various preceding formulae can thus be summarized:

$$u = -\frac{\partial\varphi}{\partial y},\quad v = \frac{\partial\varphi}{\partial x},\quad \varphi'(\rho\varphi')' + \rho w w' = 0,$$

$$\frac{2p}{u} = \frac{2q}{v} = \frac{2r}{w} = -\frac{w'}{\varphi'} = \frac{(\rho\varphi')'}{\rho w}$$

If, for example, the component of the motion parallel to the xy plane is that which is due to a rotation with constant angular velocity Ω, around the z axis, we can pose

$$\varphi = \tfrac{1}{2}\Omega\rho^2$$

and the differential relation between ϕ and w becomes

$$2\Omega^2\rho + ww' = 0,$$

From which making the integral

$$2\Omega^2\rho^2 + w^2 = Z(z, t).$$

We have definitively

$$u = -\Omega y, \quad v = \Omega x, \quad w = \sqrt{Z - 2\Omega^2\rho^2},$$

$$u^2 + v^2 + w^2 = Z - \Omega^2\rho^2,$$

$$\frac{p}{u} = \frac{q}{v} = \frac{r}{w} = \frac{\Omega}{\sqrt{Z - 2\Omega^2\rho^2}}$$

the flux lines are given by the equations

$$\rho = \text{const},$$

$$\arctan \frac{y}{x} - \int \frac{\Omega dz}{\sqrt{Z - 2\Omega^2\rho^2}} = \text{const},$$

$$t = \text{const}.$$

The motion defined by these formulae (which can be limited to a cylindrical space) is the motion of an incompressible fluid, if not when Z is independent of z: in this case, the flux lines are helical, all of which have the same z axis. These examples are sufficient to establish the existence of a large and interesting class of fluid motions, which (by an obvious analogy) can be called *helical motions*, and in which the flux lines coincide at each instant, and at each point, with the vortical lines. The necessary and sufficient conditions to define this class of motions are equations (1) and (1_a); but another form can be used that can easily be given for these equations. The first of the equations (1_a) is the following:

$$\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) w - \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) v = 0,$$

it can be written, in fact as

$$\frac{\partial u}{\partial y} v + \frac{\partial u}{\partial z} w = \frac{\partial v}{\partial x} v + \frac{\partial w}{\partial x} w,$$

and from this equation we can immediately pass to the first of the following three:

$$u' = \frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial(\omega^2)}{\partial x},$$

$$v' = \frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial(\omega^2)}{\partial y}, \qquad (4)$$

$$w' = \frac{\partial w}{\partial t} + \frac{1}{2}\frac{\partial(\omega^2)}{\partial z},$$

where u', v', and w' are the complete derivatives of u, v, w; and where, for brevity, we pose

$$u^2 + v^2 + w^2 = \omega^2.$$

These new equations, of which one is the consequence of the other two, can be considered as characteristics of each helical motion.

Now, from the well-known forms of the equations of motion of perfect fluids, we know that if the external forces have a potential function, the trinomial

$$u'dx + v'dy + w'dz$$

is a perfect differential, with respect to the coordinates, that means that the potential function for the accelerations exists. Having, from equations (4)

$$\frac{\partial w'}{\partial y} - \frac{\partial v'}{\partial z} = 2\frac{\partial p}{\partial t}, \quad \frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x} = 2\frac{\partial q}{\partial t},$$

$$\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} = 2\frac{\partial r}{\partial t},$$

it is immediately recognized that the existence of such potential functions for the accelerations cannot agree with the hypotheses of helical motion if the quantities p, q, r in this motion are independent of time. Furthermore, if we denote with μ the common value of the three ratios (1), that means we pose

$$p = \mu u, \; q = \mu v, \; r = \mu w, \qquad (4_a)$$

and if p_1, q_1, r_1, are indicated with three expressions formed with p, q, r, in the same way in which these are formed with u, v, w, we have the relationships

$$2p_1 = 2\mu p + \frac{\partial \mu}{\partial y} w - \frac{\partial \mu}{\partial z} v,$$

$$2q_1 = 2\mu q + \frac{\partial \mu}{\partial z} u - \frac{\partial \mu}{\partial x} w,$$

$$2r_1 = 2\mu r + \frac{\partial \mu}{\partial x} v - \frac{\partial \mu}{\partial y} u,$$

from which follows

$$pp_1 + qq_1, + rr_1 = (p^2 + q^2 + r^2)\mu.$$

When the quantities p, q, r and the quantities p_1, q_1, and r_1 are thus independent of time, the factor μ cannot depend upon this variable, and then, consequently, (4_a), the components of the velocity cannot also be functions only of the coordinates of position. In this way, we obtain the following theorem: When the potential of acceleration exists, then helical motion cannot be verified if this motion is not also stationary. Reciprocally, from equation (4) it follows that for each stationary helical motion there exists a potential of acceleration, a potential whose value is $\frac{1}{2}\omega^2$.

If this property of helical motion is considered, taking the ordinary derivatives of equations (4_a) with respect to x, y, and z and summing them, and denoting the density by ε, then we have

$$\mu\varepsilon' - \mu'\varepsilon = 0:$$

then: *in each stationary helical motion, the ratio between* μ *and* ε *remains constant for each fluid molecule, in the whole course of the motion.*

The equations (4) are not particular cases of the other three, which exist unconditionally. In fact, if to the second member of the equation

$$u' = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} u + \frac{\partial u}{\partial y} v + \frac{\partial u}{\partial z} w$$

we add and subtract the binomial

$$\frac{\partial v}{\partial x} v + \frac{\partial w}{\partial x} w,$$

we get the first of the following equations:

$$u' = \frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial(\omega^2)}{\partial x} + 2qw - 2rv,$$

$$v' = \frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial(\omega^2)}{\partial y} + 2ru - 2pw,$$

$$w' = \frac{\partial w}{\partial t} + \frac{1}{2}\frac{\partial(\omega^2)}{\partial z} + 2pv - 2qu,$$

from which result the equations (4), when we impose the proportionalities (1).

From the same equation (a), adding and subtracting to the second member the quantity s multiplied by u, where

$$s = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z},$$

we can also calculate the first of the following equations:

$$u' = \frac{\partial u}{\partial t} + \frac{\partial(u^2)}{\partial x} + \frac{\partial(uv)}{\partial y} + \frac{\partial(uw)}{\partial z} - su,$$

$$v' = \frac{\partial v}{\partial t} + \frac{\partial(vu)}{\partial x} + \frac{\partial(v^2)}{\partial y} + \frac{\partial(vw)}{\partial z} - sv, \qquad (5a)$$

$$w' = \frac{\partial w}{\partial t} + \frac{\partial(wu)}{\partial x} + \frac{\partial(wv)}{\partial y} + \frac{\partial(w^2)}{\partial z} - sw,$$

and the comparison with the preceding gets the following identities:

$$\frac{\partial\left(u^2 - \frac{\omega^2}{2}\right)}{\partial x} + \frac{\partial(uv)}{\partial y} + \frac{\partial(uw)}{\partial z} = su + 2qw - 2rv,$$

$$\frac{\partial(vu)}{\partial x} + \frac{\partial\left(v^2 - \frac{\omega^2}{2}\right)}{\partial y} + \frac{\partial(vw)}{\partial z} = sv + 2ru - 2pw,$$

$$\frac{\partial(wu)}{\partial x} + \frac{\partial(wv)}{\partial y} + \frac{\partial\left(w^2 - \frac{\omega^2}{2}\right)}{\partial z} = sw + 2pv - 2qu. \qquad (6)$$

when a potential of motion φ exists, we have

$$s = \Delta_2\phi, \; p = q = r = 0$$

and the preceding relationships are the well-known Maxwell formulae.

Very similar formulae also exist in the case in which the motion is without a potential but appears instead in the class of helical motions.

Taken in all of its generality, the relationships in (6) reproduce the other formulae which Maxwell calls the electromagnetic force equations (Second Edition of the *Treatise*, Vol. II, Art. 643.) In order to establish the agreement between equations (6) with Maxwell's, we must write

α, β, γ to replace $u, v, w,$
$2\pi u, 2\pi v, 2\pi w$ to replace $p, q, r,$
$4\pi m$ to replace $s,$

where α, β, γ are, according to Maxwell, the components of magnetic force; u, v, w are those of the specific intensity of the current, and m is the density of the Newtonian distribution equivalent in external action, to the magnetic polarization of the field.