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# THE SCHRÖDINGER EQUATION FROM THREE POSTULATES

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ABSTRACT
Axiomatic formulations for a physical theory are of greatest use when considering the problem of generalizing its mathematical structures; to find, restrict, or classify any possible larger theory. Historically, there has been great emphasis upon *linear operators*, and non–commuativity, as the fundamental distinguishing feature of quantum theories. Here I explore the other view, that it is *complex numbers* that are fundamental, with the remaining aspects being shown to be accidental consequences of this assumption. The result is a remarkably simple deduction of Schrödinger dynamics from three postulates. Hilbert space is a consequence of them.

## 1. Introduction

In the historical development of quantum theory it is remarkable that the views of Heisenberg[1], over what is fundamental, have so totally dominated those of Schrödinger[2]. Our present theory is, in large part, built upon the assumption that it is *operator non–commutativity* which distinguishes quantum theories from classical theories. Must it be so? Is that the *only* tenable viewpoint?

Of course, Schrödinger believed that the more important feature of quantum mechanics was its use of *complex numbers*. He tried, in vain, at an early stage, to remove them, and found that he could not[3]. Therefore, while his view is considered *neo–classical* nowadays, and therefore something to be despised, Schrödinger would surely have considered this observation to strike at the *essence* of the change made over the preceding ideas. Whereas classical physics employed partial differential equations—their use of complex numbers was mere artifice. The initial success of his equation confirmed, Schrödinger was surprised, and rather worried, that complex numbers appeared *essential!* Such a personal experience, his great doubt resolved, might *explain* his dissenting views about what constitutes a *quantum* theory[4].

From the mathematical viewpoint, non–commutativity is unsurprising. Since the work of Lie, one knows it will be ubiquitous in the study of systems of equations, first–order in time, that admit Lie symmetries[5]. What is special, however, are those symmetries associated with systems of equations that employ *complex numbers*. It is no trivial matter to construct a complex–valued dynamical system.

In algebra complex numbers are *more general* than real numbers; in geometry they are *far more special*. To introduce $i$ is to introduce a notion of orthogonal real and imaginary parts, upon an pair of reals. The mappings, in time, that are the solutions of a dynamical system, must preserve the complex structure, if not in absolute degree, then in some sensible mathematical relaxation of it.

This fact is well–known to the mathematicians, in their studies of the differential geometry, and taxonomy of *almost complex manifolds*[6]. To see it in a physical context, we will exhibit Schrödinger mechanics, and Hilbert space geometry, as a *complex restriction* of the mathematics of Hamilton, as used in the classical point mechanics. It will *not be* the trail blazed by Schrödinger[2]; to use *analogies* with the classical Hamilton–Jacobi theory. Rather we will deduce the dynamical equations directly from the *classical mathematics* and two postulates of restriction[7].

The object is to demonstrate that we can *just as well* take complex numbers as the basic change. The point is to find a new path to: *generalize by restriction*. In a nut–shell, we look for an enclosing formalism for quantum theory, and ask how it is to be made compatible with the complex numbers.

## 2. The specialness of bilinear hermitian forms

Expectation values are the basis of quantum predictions, including transition probabilities. Yet these quantities must involve states, and operators, in combination. Even if we first solved a dynamical problem in the Heisenberg picture, one must choose a state vector to fully specify a solution. For the familar problem

$$i\hbar\frac{\partial\psi(\mathbf{x},t)}{\partial t} = \left\{-\frac{\hbar^2}{2m}\nabla^2_{\mathbf{x}} + V(\mathbf{x},t)\right\}\psi(\mathbf{x},t), \tag{1}$$

as with other non–relativistic examples, we may abstract the equation

$$i\hbar\frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle, \tag{2}$$

with $\hat{H}$ an hermitian operator. Choosing a complete set of states $|k\rangle$, we may expand $|\psi\rangle = \psi_k|k\rangle$, with $\psi_k = \langle k|\psi\rangle$, so that the expectation value is written

$$h(\psi,\bar{\psi}) = \langle\psi|\hat{H}|\psi\rangle = \bar{\psi}_j H_{jk}\psi_k, \tag{3}$$

where $H_{jk} = \langle j|\hat{H}|k\rangle$, and the summation convention is implicit. As elegant as the bra–ket notation of Dirac may be, we could just as well write

$$i\hbar\frac{d\psi_j}{dt} = H_{jk}\psi_k = \frac{\partial h(\psi,\bar{\psi})}{\partial\bar{\psi}_j}, \tag{4}$$

and use this as the Schrödinger equation. In this picture it is natural to ask, as Weinberg has already done[8], whether the expectation values of quantum theory must *always* be hermitian forms, and, further, whether they must have *constant* coefficients $H_{jk}$. For instance, what of[9,10,11,12]

$$i\hbar\frac{d}{dt}|\psi\rangle = \hat{H}(\psi,\bar{\psi})|\psi\rangle, \tag{5}$$

as a possibility. With thoughts patterned linear, one might think this equation ill–defined, until we recall the Hartree–Fock approximation[13]

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x}_1,\ldots,\mathbf{x}_N,t) = \left\{-\sum_{i=1}^{N}\frac{\hbar^2}{2m_i}\nabla^2_{\mathbf{x}_i} + \sum_{i,j;i\neq j}^{N}\frac{e_ie_j}{2}\int_{\mathbf{R}^3}\frac{\rho_j(\tilde{\mathbf{x}}_j)}{|\mathbf{x}_i-\tilde{\mathbf{x}}_j|}\,\mathrm{d}^3\tilde{\mathbf{x}}_j\right\}\psi, \quad (6)$$

with $\rho_j(\mathbf{x}_j)$ the integrated one–particle density.

My point is; even in the linear theory we are often using nonlinear equations formulated on complex fields. There *must* be a mathematical theory for them, and it will not be a trivial thing. Complex nonlinear dynamics is ubiquitous in practical physics via those *quantum approximations* which introduce decorrelation[13,14].

## 3. Hamiltonian mechanics

Let us recall the basic notions behind the classical equations of Hamilton[15]. One fixes upon a phase space of dimension $f$, with the generalized coordinates $q_j$ and momenta $p_j$, where $j \in [1, f]$. We then consider the dynamical system

$$\frac{d}{dt} \equiv \{\bullet, h\}_{\mathrm{PB}}, \quad (7)$$

where $h(\mathbf{q},\mathbf{p}) = h(q_1,\ldots,q_f,p_1,\ldots,p_f)$ is the hamiltonian function and

$$\{\bullet,\bullet\}_{\mathrm{PB}} = \frac{\partial}{\partial q_j}\frac{\partial}{\partial p_j} - \frac{\partial}{\partial p_j}\frac{\partial}{\partial q_j} \quad (8)$$

is the Poisson bracket (summation convention implicit). Here we allow that $f$ might take any value whatsoever, and that $h$ might be any sufficiently well–behaved function. In the sequel we will restrict $h$.

This system of equations, if linearized about any point, exhibits an obvious symmetry group, the well–known *symplectic group*, $Sp(2f)$, in $f$ dimensions[15]. It is basic to the area–preserving nature of a classical flow, and its higher integral invariants. Other symmetries, such as those which are associated with preserved lengths, or angles, the *orthogonal group*, $O(2f)$, have no place in this most general scheme, except via restricted choices as to the function $h$ (harmonic oscillators).

## 4. Quantum dynamics as complexified classical dynamics

Suppose now that we allow a different view of the quantities $q_j$ and $p_j$. From the mathematical standpoint they are simply two sets of real numbers, associated in pairs. We do not *have* to identify them as particle coordinates. On this basis, Strocchi[16], and Heslot[17] have introduced complexified classical coordinates:

$$\psi_j = (q_j + ip_j)/\sqrt{2}, \quad (9)$$

$$\bar{\psi}_j = (q_j - ip_j)/\sqrt{2}; \quad (10)$$

from which we obtain the *complexified Poisson bracket*

$$\{\bullet,\bullet\}_{\tilde{\mathrm{PB}}} = \frac{\partial}{\partial \psi_j}\frac{\partial}{\partial \bar{\psi}_j} - \frac{\partial}{\partial \bar{\psi}_j}\frac{\partial}{\partial \psi_j} = \{\bullet,\bullet\}_{\mathrm{PB}}/i, \quad (11)$$

and the equivalent equation of motion

$$i\frac{d}{dt} \equiv \{\bullet, h\}_{\tilde{\mathrm{PB}}}. \tag{12}$$

Strocchi[16], Heslot[17] and Weinberg[8], among others, have observed that by choosing only those $h$ which are *bilinear hermitian forms*, i.e.

$$h(\psi, \bar{\psi}) = \bar{\psi}_j H_{jk} \psi_k, \tag{13}$$

with $\bar{H}_{jk} = H_{kj}$ (hermiticity), we find that

$$\{g, h\}_{\tilde{\mathrm{PB}}} = \bar{\psi}_j G_{jk} H_{kl} - H_{jk} G_{kl} \psi_l \tag{14}$$

$$\{\psi_j, h\}_{\tilde{\mathrm{PB}}} = H_{jk}\psi_k. \tag{15}$$

Employing now the notations $\psi_j = \langle j|\psi\rangle$, and $H_{jk} = \langle j|\hat{H}|k\rangle$, one makes contact with the usual Dirac formulation of quantum theory.

As such, it is possible to embed the two quantum equations

$$i\hbar\frac{d\langle\hat{G}\rangle}{dt} = \langle\psi|[\hat{G}, \hat{H}]|\psi\rangle \tag{16}$$

$$i\hbar\frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle \tag{17}$$

within the Hamiltonian formalism, by complexification.

Since complex coordinates have no physical dimension it is not possible to carry over any classical analogy with the generalized coordinates and momenta. Evidently, such a mathematical system demands a different physical interpretation. The only thing one may carry over is the identification of $h$, the dynamical generator, as a real–valued quantity with dimensions of energy. If that is to be done, then we need to insert a constant $\hbar$, with dimensions of action, upon the left–hand side.

This system of equations, being *already linearized*, exhibits an obvious symmetry group, the well–known *unitary group*, $U(f)$, in $f$ dimensions. This is basic to the Copenhagen interpretation of quantum theory. There one simply *postulates* the rule for calculating transition probabilities. If one is to make such a postulate, and keep it independent of how we choose our complex coordinates, then Wigner's theorem confines attention to linear unitary, and anti–linear, anti–unitary symmetries[18]

## 5. Specialization or Generalization?

The preceding is well–known. Mostly, it has not been paid much attention by physicists. Perhaps this may be because it is unclear whether the introduction of *complex coordinates*, represents a *generalization*, or a *specialization*.

The point is, if it were a specialization, then we would *know* how to find a generalized quantum dynamics! We would know a larger mathematical structure—the Hamiltonian formalism; and a sieve, a quantum physical principle—at all costs, make this consistent with complex numbers.

## 6. Conformal mappings, orthogonality and analyticity

Long ago, Strocchi[16] saw the importance of analyticity in the deeper elucidation of quantum dynamics, and its connection with classical dynamics. He observed that it is a *specialization*. As a precognition, note the group identity[15]

$$U(f) = Sp(2f) \cap O(2f). \tag{18}$$

Unitary geometry is a restriction of symplectic geometry. We must ask: Why?

At root, complex geometry is intimately related to the orthogonal group. When we introduce an imaginary unit $i$, we have chosen to keep in our sight a notion of *orthogonal components*. That is what $i = e^{i\pi/2}$ means; a rotation, by $\pi/2$, from which we derive a multiplication, being that of the *orthogonal group*.

When considering infinitesimal mappings of the complex plane into itself, it is an old and classic problem to classify the angle–preserving transformations. The result is the *conformal mapping theorem*[19]. It says that the only *isogonal*, i.e. locally and strictly angle–preserving mappings have analytic generators. To the general map $z \mapsto w(z, \bar{z})$, one applies the Cauchy–Riemann conditions, to get $\partial_{\bar{z}} w = 0$. For dynamics, i.e. $\dot{z} = w(z, \bar{z})$, we have the analyticity constraint

$$\partial_{\bar{z}} \dot{z} = \partial_{\bar{z}} w(z, \bar{z}) = 0 \text{ and c.c.}, \tag{19}$$

i.e. we apply Cauchy–Riemann conditions to the *time–derivative*[7].

## 7. The classical imaginary unit

If we are to introduce $i$, without smuggling it in, then we must identify a quantity that can play the role of $\sqrt{-1}$. Return now to ordinary classical dynamics, and introduce, in the manner of Arnold[15], a vectorial notation

$$\mathbf{z}_{2f} = \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix}, \quad \text{where } \mathbf{q} = (q_1, \ldots, q_f), \text{ and } \mathbf{p} = (p_1, \ldots, p_f),$$

along with the matrix

$$\mathbf{I}_{2f} \equiv \begin{pmatrix} 0 & -\mathbf{E}_f \\ +\mathbf{E}_f & 0 \end{pmatrix} \tag{20}$$

where $\mathbf{E}_f$ is the $f \times f$ identity matrix. Then, $\mathbf{I}_{2f}^2 = -\mathbf{E}_{2f}$, so that $\mathbf{I}_{2f}$ is certainly the square root of minus the identity. Further, denoting the matrix transpose by a superscript $T$, we find $\mathbf{I}_{2f}^{\mathrm{T}} = -\mathbf{I}_{2f} = \mathbf{I}_{2f}^{-1}$.

Thus we have: an imaginary unit $i$, $\mathbf{I}_{2f}$; a complex conjugate operation $*$, which is $T$ of the units $\mathbf{I}_{2f}$ and $\mathbf{E}_{2f}$; and an hermitian adjoint operation $\dagger$, which is $T$ of the entire complexified vector. Explicitly,

$$\mathbf{z}_{2f} \equiv \mathbf{E}_{2f} \begin{pmatrix} \mathbf{q} \\ \mathbf{0} \end{pmatrix} + \mathbf{I}_{2f} \begin{pmatrix} \mathbf{p} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} +\mathbf{q} \\ +\mathbf{p} \end{pmatrix} = \mathbf{q} + i\mathbf{p}, \tag{21}$$

$$\mathbf{z}_{2f}^{*} \equiv \mathbf{E}_{2f}^{\mathrm{T}} \begin{pmatrix} \mathbf{q} \\ \mathbf{0} \end{pmatrix} + \mathbf{I}_{2f}^{\mathrm{T}} \begin{pmatrix} \mathbf{p} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} +\mathbf{q} \\ -\mathbf{p} \end{pmatrix} = \mathbf{q} - i\mathbf{p}, \tag{22}$$

$$\mathbf{z}_{2f}^{\dagger} \equiv \left\{ \mathbf{E}_{2f} \begin{pmatrix} \mathbf{q} \\ \mathbf{0} \end{pmatrix} + \mathbf{I}_{2f} \begin{pmatrix} \mathbf{p} \\ \mathbf{0} \end{pmatrix} \right\}^{\mathrm{T}} = (+\mathbf{q} \quad +\mathbf{p}). \tag{23}$$

With these definitions, multiplication by a complex scalar $a+ib$ is defined

$$(a+ib)\mathbf{z}_{2f} = a\mathbf{E}_{2f}\begin{pmatrix}\mathbf{q}\\ \mathbf{p}\end{pmatrix} + b\mathbf{I}_{2f}\begin{pmatrix}\mathbf{q}\\ \mathbf{p}\end{pmatrix} = (a\mathbf{q}-b\mathbf{p}) + i(a\mathbf{p}+b\mathbf{q}), \tag{24}$$

and the induced hermitian inner–product reads

$$\mathbf{w}_{2f}^{\dagger}\mathbf{z}_{2f} = (\mathbf{q}_w\cdot\mathbf{q}_z + \mathbf{p}_w\cdot\mathbf{p}_z) + i(\mathbf{q}_w\cdot\mathbf{p}_z - \mathbf{p}_w\cdot\mathbf{q}_z) \in \mathbf{C}, \tag{25}$$

where we *must* multiply things out keeping track of the units $\mathbf{E}_{2f}$ and $\mathbf{I}_{2f}$.

In this way, one equips a real vector space of dimension $f$ with the operations of complex algebra. The quantity $\mathbf{I}_{2f}$ is called the *complex structure tensor* (here in canonical form). The differential geometry of complex, and almost complex manifolds[6] develops the usual assumption that $\mathbf{I}_{2f}$ need not be a constant (i.e. basis vectors may vary). Just as symplectic differential geometry underlies real–valued Hamiltonian dynamics[15], the almost complex differential geometry must underlie complex–valued dynamical systems.

## 8. The conformal restriction of Hamilton

Let us now apply the Cauchy–Riemann conditions to the equations of Hamilton. We introduce an $i$, assume it is a *constant*, and demand that

$$i\dot{\psi}_k = i(\dot{q}_k + i\dot{p}_k)/\sqrt{2} = \left(\frac{\partial h}{\partial q_k} + i\frac{\partial h}{\partial p_k}\right)/\sqrt{2}, \tag{26}$$

is analytic. Applying the operator

$$\frac{\partial}{\partial\bar{\psi}_j} = \left(\frac{\partial}{\partial q_j} + i\frac{\partial}{\partial p_j}\right)/\sqrt{2},$$

to both sides, we set $\partial_{\bar{\psi}_j}[\dot{\psi}_k] = 0 + i0$, and obtain the conditions

$$\frac{\partial^2 h}{\partial q_j\partial p_k} + \frac{\partial^2 h}{\partial p_j\partial q_k} = 0, \tag{27}$$

$$\frac{\partial^2 h}{\partial p_j\partial p_k} - \frac{\partial^2 h}{\partial q_j\partial q_k} = 0, \tag{28}$$

(for all $j,k\in[1,f]$). Any function satisfying Eq. (27) and Eq. (28) will generate a complex analytic dynamics, via the standard equations of Hamilton.

Since they are second–order identities, we substitute a general second–order power series, equate coefficients, and obtain the solution

$$h(\mathbf{q},\mathbf{p}) = a + b_jq_j + c_jp_j + \frac{1}{2}\left(A_{jk}q_jq_k + 2B_{jk}p_jq_k + A_{jk}p_jp_k\right), \tag{29}$$

where all coefficients are real–valued, and we need: $B_{jk} = -B_{kj}$ to satisfy Eq. (27); and $A_{jk} = A_{kj}$, from Eq. (28), since the partials commute. Now we compute

$$\frac{\partial h}{\partial q_k} = b_k + A_{kl}q_l - B_{kl}p_l \tag{30}$$

$$\frac{\partial h}{\partial p_k} = c_k + A_{kl}p_l + B_{kl}q_l. \tag{31}$$

In vectorial notation, the final result reads

$$\mathbf{I}_{2f}\frac{d}{dt}\begin{pmatrix}\mathbf{q}\\ \mathbf{p}\end{pmatrix}=\begin{pmatrix}\mathbf{b}\\ \mathbf{c}\end{pmatrix}+\begin{pmatrix}\mathbf{A} & -\mathbf{B}\\ \mathbf{B} & \mathbf{A}\end{pmatrix}\begin{pmatrix}\mathbf{q}\\ \mathbf{p}\end{pmatrix} \tag{32}$$

Introducing the matrices:

$$\psi_0 \equiv (\mathbf{b}+i\mathbf{c})/\sqrt{2}, \tag{33}$$

$$\mathbf{H} \equiv (\mathbf{A}_{2f}+i\mathbf{B}_{2f})/\sqrt{2}; \tag{34}$$

we can rewrite this as

$$i\frac{d\psi}{dt}=\psi_0+\mathbf{H}\psi, \tag{35}$$

where, since $\mathbf{A}_{2f}^{\mathrm{T}}=\mathbf{A}_{2f}$, and $\mathbf{B}_{2f}^{\mathrm{T}}=-\mathbf{B}_{2f}$, we have

$$\mathbf{H}^{\dagger}=\mathbf{A}_{2f}^{\mathrm{T}}+[i\mathbf{B}_{2f}]^{\mathrm{T}}=\mathbf{H}, \tag{36}$$

so that the generating operator is hermitian self–adjoint.

Having shown the above, in the formalism of complexified classical dynamics, we may now revert to the usual complex notation. Then Eq. (27) and Eq. (28) assume the more symmetric form

$$\frac{\partial^2 h}{\partial\psi_j\partial\psi_k}=0, \quad \text{and c.c.} \tag{37}$$

for all $j,k\in[1,f]$. Notice that *non–commuting operators* appear by restriction. One has to narrow down the class of functions $h$ to obtain them. Less is more.

## 9. The projective restriction of Hamilton

To remove the constant factor $\psi_0$, in a physically meaningful way, we invoke an idea of Kibble[9], and demand the scaling invariance: $\psi(t)\mapsto\lambda\psi(t)$. A more elegant statement is the Weinberg homogeneity constraint[8]

$$h(\lambda\psi,\bar{\psi})=\lambda h(\psi,\bar{\psi})=h(\psi,\lambda\bar{\psi}). \tag{38}$$

Differentiation against $\lambda$ reduces this to the identities

$$\psi_j\frac{\partial h}{\partial\psi_j}=h=\bar{\psi}_j\frac{\partial h}{\partial\bar{\psi}_j}. \tag{39}$$

Applying this condition, along with Eq. (37), we obtain, uniquely, the functions $h(\psi,\bar{\psi})=\bar{\psi}_j H_{jk}\psi_k$, where reality demands that $\bar{H}_{jk}=H_{kj}$.

## 10. The origin of Hilbert space

The Hilbert space geometry is reconstructed by applying spectral theory to the square of the generator $\hat{H}$[7]. This gives us a positive–definite matrix (we allow no zero energies). As is well–known, any such hermitian form provides an inner product, and, via the constancy of its coefficients, this inner–product is seen to be an invariant. Thus we obtain an Hilbert space geometry, and we can recover the traditional language of the Dirac transformation theory.

Moroever, since $f$ is arbitrary, we have covered all possible unitary symmetries $U(f)$. Their realizations upon Hilbert spaces of all possible dimension, which carry representations of any chosen abstract group, will generate special cases of the given structure. The only thing one cannot obtain a *priori* is the usual statistical interpretation. Of course, at the present time, this is just a postulate anyway.

## 11. The three postulates

Thus the combination[7]:

$$\begin{aligned} \mathcal{P}_1 &: \quad \text{Hamiltonian formalism} \\ \mathcal{P}_2 &: \quad \text{Complex structure} \\ \mathcal{P}_3 &: \quad \text{Projective structure} \end{aligned}$$

is sufficient to deduce the Schrödinger dynamics, for all possible $f$. Unlike the usual axiomatization[20], Hilbert space is now a *consequence*.

The given axioms are new, so far as I know, but the observation that *complex projective structure*, and *Kähler geometry* underlie linear quantum dynamics has been widely appreciated. See, e.g. Cirelli et al.[21]. The main reason I have sought a new statement is to formulate the postulates as differential identities: Eq. (37), and Eq. (39). Then, and only then, can we try relaxing the axioms, *in degree*, e.g.

$$\frac{\partial^2 h}{\partial\psi_j \partial\psi_k} \neq 0. \text{ and c.c.} \tag{40}$$

Since *zero* is precisely the usual linear theory, a “small” value ought to mean weakly nonlinear, and a “big” value, strongly nonlinear. The value of this axiomatization is thus to identify departures from linearity with *non–analyticities*.

## 12. Nonlinearity and non–analyticity

We must now investigate if non–analytic complex coordinate transition functions make any physical sense. Interestingly, the projective postulate, as expressed by Eq. (38), enforces the general decomposition

$$h(\psi, \bar{\psi}) = \bar{\psi}_k \frac{\partial^2 h}{\partial\bar{\psi}_k \partial\psi_j} \psi_j. \tag{41}$$

Moreover, as Weinberg has shown[8], such functions form a Lie algebra in complexified Poisson bracket. Thus we retain, in general, a notion of operators, but these are now more like tensors. At the very least, they are *form–invariant*. Further, we can always single out one such hermitian form, obtain a local inner–product, and thus erect a *tangent Hilbert space*. There is no longer a global linear structure; it is now a purely local notion applicable only to infinitesimal transformations.

This gives a clue to the meaning of non–analyticity. In the theory of Berry phases one considers evolution under an adiabatically cycled Hamiltonian[22]. There we find that a *geometric* phase factor is required, being given by an hermitian connection in the projective space of states. It is non–analytic, and non–integrable, being related to the parallel transport of eigenvectors in Hamiltonian space. Guided by this idea we may explore the generalized gauge transformations[7]

$$\psi(t) \mapsto \tilde{\psi}(t) \equiv e^{i\Phi(\psi,\bar{\psi};t)}\psi(t).$$

with $\Phi(\psi, \bar{\psi}; t)$ arbitrary. These leave functions of the type Eq. (38) invariant, but they destroy analyticity, since $\dot{\tilde{\psi}}_k(t) = e^{i\Phi}\dot{\psi}_k(t) + i\{d_t\Phi\}e^{i\Phi}\psi_k(t)$. The mapping is only analytic: *up to phase*. Thus it seems that we can allow Eq. (40) and that non–analyticity is connected with a generalized gauge freedom[23,24,25]

## 13. Conclusion

In summary, we have shown that ordinary linear quantum dynamics can be viewed as the conformal and projective restriction of Hamiltonian mechanics. The non–commuting linear operators appear as a *new* layer of extra structure peculiar to this restriction. The lesson lies in the specialness of complex numbers. Perhaps this is a useful guiding principle: *Let there be a complex dynamical system*. Can we say what it *must* look like, if it is to be a Hamiltonian system? Is it just Weinberg[8]?

On a practical level, the reason to be interested in this question is simply to develop the mathematics of *quantum approximations*. Almost never do we use the quantum theory in pristine text–book–linear form. One must generally simplify equations by treating some parts of the system classically. Yet, as I have shown elsewhere[11], this induces a nonlinearity via the replacement of some operators by their $c$–number expectations. The value of a larger system of mathematics is that we can do this computation within a single unified formalism.

On a fundamental level, suspension of the superposition principle only makes sense for quasi–classical physics. Nowadays, it is not uncommon to break with the tradition of Heisenberg[1], and suggest that we need a unified micro and macro physics[26]. Presumably, this means: *we are no longer of the opinion that $\psi$ is simply the probability of finding a point–like particle here or there.* Then we require another meaning, and *recovery* of the standard statistical rules, as a consequence.

It is for this reason that I have argued that a fundamental nonlinear theory must solve the measurement problem[7]. The recent research has taught us that the usual statistical rules will not carry over as *postulates*[8,12]. Therefore, one would *have* to treat measurements as having a *physics* attached to them. Here the aim would be to complete the physics of decoherence[28], i.e. to invoke an emergent, and necessarily weak[27], quasi–classical nonlinearity that would reduce the decohered density matrix to just one alternative, in the proper statistical fashion.

Perhaps the mathematics can tell us more about these questions. For instance, it might provide clues about how to do it; or good reasons why such a program will never work[29]. In any case, I find complex nonlinear dynamics fascinating.

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