Notes on the Mathematical Theory Of Electrodynamic Solenoids

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The solenoids considered in this note are continuous systems of currents in which the law of distribution is much more general than is ordinarily admitted.

I assume that they are formed in the following way:

Take a closed space, S, which is bounded in every direction by a closed surface, ω. If a function ϕ $(x, y, z,)$ which has the coordinates of the points of S is assumed, the only condition on this function is, that in this space it is single-valued, finite, and continuous, and so are its partial derivatives. The surfaces represented by the equation ϕ = constant, which I will indicate with the symbol (ϕ), are all distinct from each other, and the lines, in which they intersect the surface ω, are also distinct from each other and closed. These lines divide the surface, ω, into an infinity of elementary strips, each of which corresponds to the infinitesimal increment $d\phi$ of the function ϕ. Then the current is made to circulate in each of these strips in a determinate sense. The intensity of the current is κ $d\phi$, where κ is a constant and $d\phi$ is the increment of the function ϕ passing from one strip to the other. All of these currents form the general solenoids that I wish to study at this time.

It will be useful at this point to review some general information about this system. First, note that the way in which the system is formed does not exclude that the geometrical locus of the effective currents can be an open surface rather than one which is closed. In fact, if we make one part of the ω surface coincide with a portion of one of the (ϕ) surfaces, the boundary line of this common portion becomes a terminal current of the solenoid because the intensity of the currents is zero, which in the general case would circulate on the common portion.

Second, if we suppose the κ factor to be constant, we do not thereby restrict the law by which the intensity varies. In fact, if we want to make κ change

Editor's Note

Δ_1 and Δ_2 are differential operators of the first and second degree or kind. These are defined in a paper entitled *Richerche di analizi applicate alla Geometrie* in his *Collected Works*, Vol. 1, page 143. Beltrami's differential operators may be expressed in terms of Einstein's notation as follows:

$$\Delta_1 f = g^{ij} f_{,i} f_{,j} = g^{ij} (\partial f / \partial x^i)(\partial f / \partial x^j) \qquad (1)$$

This is the Beltrami operator of the first kind that assigns to each scalar field f the length of the gradient of f, grad f, or ∇f.

$$\Delta_2 f = -g^{ij} f_{,i,j}\, \partial^2 / (\partial x^i \partial x^j) - g^{ij} \Gamma^k_{ij}\, \partial / \partial x^k \qquad (2)$$

This operator is of the second kind and is identical with $-\Delta$ where Δ denotes the Laplacian:

$$\nabla^2 f = \text{div grad } f = \nabla \cdot \nabla f =$$
$$1/\sqrt{g}\, \partial [\sqrt{g}\, g^{ij} (\partial f / \partial x^j)] / \partial x^i$$

from one strip to the other, for example, if $\kappa = F(\phi)$, it will be sufficient to substitute for ϕ a function $\Psi = \int F(\phi) d\phi$. In such a case, there would be a new system substantially equivalent to the first, in which the intensity of each of the elementary currents will be expressed $d\Psi$. In order to simplify, I will suppose that $\kappa = 1$, since multiplying the final result by κ if it reenters the first hypothesis.

Furthermore, I can say that the necessity for the condition that requires the derivative of the function ϕ to be single-valued does not imply that the function must necessarily be single-valued as well. It is useful to be able to take a function that is not single-valued (this means that it has periodic form) in order to apply this method of the solenoids to more general cases; but, in this case, I will consider the ϕ functions as single-valued in order to simplify the research on the potential function of the solenoids, which is almost intuitive, and at the end of the note, I shall give a second demonstration of the formula derived, from which it will be very easy to study the value of functions which are not single-valued.

For greater clarity, I shall consider the hypothesis that the successive lines of intersection of the ω surfaces (ϕ) correspond to the values of the preceding ϕ parameter, from a minimum value ϕ_1 to a maximum value ϕ_2. In fact, if the ω surface is not consistent with this hypothesis, we can change it by substituting closed surfaces that do satisfy these conditions, and in this way we can add some diaphragms that have circulating currents, or portions of currents of equal intensity on each of their two faces, that move in opposite directions.

I denote, with n, the direction of the inwardly facing normal to an element of the surface $d\omega$, and with r, the distance of this element, which has x, y, z as coordinates, from a point m_1, which has as coordinates x_1, y_1, and z_1, on which we suppose the electromagnetic action of the solenoids to act. I consider that this last point is at a finite distance from the ω surface, from the internal to the external part.

Now consider one of the elementary strips in which the ω surface is divided, and let ϕ, and ϕ plus $d\phi$ be the values of the parameter corresponding to the two lines that have this strip between them. I will suppose the increment $d\phi$, to be positive, of the pair of all the analogous increments; it is not necessary that the rest of these increments be equal. Let ω' be the portion of ω that is the locus of all the lines (ϕ) in which the parameters are between the intermediate value ϕ and the maximum value ϕ_2. This portion of the surface is totally determined by the closed line of the ϕ parameter; therefore, because of Ampere's fundamental theorem on the electromagnetic action of closed currents, the potential on a point m_1, of the current with intensity $d\phi$, which circulates around the surface ω', can be expressed by the integral

$$d\varphi \int \frac{d\frac{1}{r}}{dn'}\, d\omega',$$

extended to all the surfaces ω'. Now the potential Φ of the entire solenoid is evidently the sum of all the expressions like this, relative to the successive increments of ϕ, from ϕ_1 to ϕ_2. Summing these, and collecting all the factors that are to be multiplied, for each element $d\omega$ of the total surface, we have

$$\Phi = \int \left(\frac{d\frac{1}{r}}{dn} \sum d\varphi \right) d\omega.$$

Then it is clear that

$$\Sigma d\phi = \phi - \phi_1,$$

where ϕ is the value of the parameter on the element $d\omega$: then

$$\Phi = \int \varphi \frac{d\frac{1}{r}}{dn}\, d\omega - \varphi_1 \int \frac{d\frac{1}{r}}{dn}\, d\omega,$$

that means, according to a well-known theorem

$$\Phi = \int \varphi \frac{d\frac{1}{r}}{dn}\, d\omega - 4\pi\epsilon\varphi_1,$$

where ε is 1 or 0, according to whether the point m_1 is internal or external to the space S,

The function Φ so determined is continuous in all the space. But it is much more useful to replace this function (which I will continue to denote with the same symbol)

$$\Phi = \int \varphi \frac{d\frac{1}{r}}{dn}\, d\omega, \tag{1}$$

which has the same derivatives of the preceding, even if it is discontinuous in all the points of the ω surface. This last circumstance is not influential, because, as I said, the point m_1, need never cross this surface.

We remember now Green's equation

$$\int \left(\varphi \frac{d\frac{1}{r}}{dn} - \frac{1}{r}\frac{d\varphi}{dn} \right) d\omega = \int \frac{\Delta^2 \varphi}{r} dS + 4\pi\epsilon\varphi(x_1, y_1, z_1),$$

where ε has the same meaning as before, and where we pose

$$\Delta^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} .$$

If for brevity these notations are used

$$P = \int \frac{\Delta^2\varphi}{r} dS, \quad \Pi = \int \frac{d\varphi}{dn} \frac{d\omega}{r},$$

we will have

$$\Phi = P + \Pi + 4\pi\varepsilon\phi_1, \tag{2}$$

where ϕ_1 represents the value of the function ϕ at the point m_1.

The two functions P and Π are two Newtonian potentials, one of space, the other of a surface that corresponds to the well-known distribution of matter in the space S and on the surface ω, distributions that I shall call p and π. The total masses of these two distributions are equal in their absolute value, but contrary in sign, because of the equation

$$\int \Delta^2\varphi \cdot dS + \int \frac{d\varphi}{dn} \cdot d\omega = 0.$$

Of the two expressions (1) and (2) of the potential Φ, the second is, in the greater number of cases, far more useful than the first one. Therefore we will not speak about the significance of equation (1), but only about the results contained in equation (2):

The electromagnetic action of the general solenoid Σ, is equal to the resultant Newtonian action at each point external to S, that is, due to the distribution p and π; and, in each internal point, the resultant action is not only due to the distribution of these two actions and to a third, whose potential is $4\pi\phi$.

This theorem is true without modification also in the case in which ϕ is a multivalued function with a single-valued differential function, provided that the branching lines are external to space S. This results immediately from the demonstration that I will give of equation (1), since all that I am about to say refers to the multi-valued functions, it is understood with the restriction or so-called branch lines. Furthermore, because it is possible to satisfy the conditions of the (LaPlacian equation) $\Delta^2\phi = 0$, in the majority of the cases, and, because it is really true in each case which we know today of the general theorem that we enunciated previously, I shall suppose that the condition be true. That means that we suppose that the distribution p be 0, then $P = 0$, then we must consider finally the distribution π, with the surface potential Π relative to π.

The simplest way to satisfy the equation $\Delta^2\phi = 0$ is to take as ϕ a linear function of the coordinates x,y,z. In this case, the solenoid Σ is formed by currents placed in parallel planes, and with the intensity constant, if the distance between planes is constant. With these hypotheses, the previous theorem reproduces what is called the Riecke theorem.[1] The observation that the author Riecke made, about the possibility of substituting the surface distribution π, which has a variable density, a certain space distribution, with constant density is nothing but a corollary of another general property about which I recommend that the reader refer to my monograph "Research on Fluid Kinematics"[2] because this theorem is related to the motion of a fluid.

Second, suppose that the surface ω be formed by a tubular portion, where the axis is orthogonal to the surface (ϕ) [at each point], and where it is closed at its two ends by the intersection of two surfaces. In this case, the distribution π exists only on the two terminal portions of the tube because on all the tubular part we have $d\phi/dn = 0$. The electric currents circulate only around the tubular portion, and the theorem describing the action of these currents is the same as the theorem of Lipschitz.[3]

These two theorems, by Riecke and Lipschitz, have a particular case in common in which they agree. It is when ϕ is a linear function and ω is a cylindrical surface whose axis is perpendicular to the planes ϕ = constant, and whose ends are closed by terminal sections that coincide with two of these planes. The theorem about this cylindrical solenoid with an arbitrary axis was already given a long time ago by F. Neumann[4] and was recently discussed with great accuracy by Emilio Weyr.[5]

A case that is in some ways reciprocal to Neumann and Weyr is as follows: If ϕ is a cylindrical potential, ϕ_1 and ϕ_2 the parameters of two equipotential surfaces outside of the acting mass, and ϕ_1 is internal to ϕ_2; and lastly, let ρ' and ρ'' be the portions that are cut by these surfaces. Applying the general theorem to the surface ω that is formed by the two plane bases ρ' and ρ'' of the cylindrical portions ϕ_1 and ϕ_2, it is clear that the distribution π exists in this case only on the latter two, and the electrical current circulates around the two former; and therefore the actions of these are equivalent for external points; however, they differ at internal points, because of the action of the potential ϕ. Suppose, for example, that $\phi = \log r$, where r is the distance between the point (x, y, z) and a given straight line, which means that the cylindrical potential must be the same as that of an infinite straight line; and let the radii of the two cylindrical surfaces ϕ_1 and ϕ_2 be a_1 and a_2. Because a_1 is less than a_2 we have

$$\frac{d\varphi}{dn} = \frac{1}{a_1} \qquad \text{on the surface } \varphi_1,$$

$$\frac{d\varphi}{dn} = -\frac{1}{a_2} \qquad \text{on the surface } \varphi_2,$$

and the potential of the electrical currents that circulate around the two rings, ρ' and ρ'', are given by

$$\Pi = \frac{1}{a_1}\int \frac{d\omega_1}{r_1} - \frac{1}{a_2}\int \frac{d\omega_2}{r_2} + 4\pi\epsilon \log r,$$

where the indices 1,2 are used to distinguish between the relative quantities of the two cylindrical portions ϕ_1 and ϕ_2. The two integrals found in this formula are evidently the Newtonian potentials for two strata, having the density of 1, that are deposited on the two cylindrical surfaces. The electric current circulates around the two rings ρ' and ρ'', which are concentric circles at the two boundaries, with an intensity inversely proportional to the respective radii, supposing that the width of the strip be constant. If we put one of these base planes, for example, ρ'', at infinity, we are left with only ρ', having two electric currents, and preceding formula reproduces the results of Emilio Weyr, arrived at in a different way, published in Volume 13, in *Schlomilch's Journal* (1868) p. 437. The general theorem explains, at the same time, the difference that occurs in the electromagnetic action, between the case where the point at which the electromagnetic action is acting is projected inside the rings and that in which it falls outside.

I will suppose now that the surface ω be orthogonal at each point to (ϕ), which cuts the surface. Generally speaking, that means that ϕ must be a multivalued potential function, and that, in this case, the form of the surface ω is like that of a tube which goes inside of itself. With this hypothesis we have $P = 0$, and $\Pi = 0$, and we have

$$\Phi = 4\pi\varepsilon\phi,$$

which means that: the external action of the solenoid built in such a way is 0, while the internal action is given by the product of 4π and the actions of the external currents that produce the potential ϕ. I regard the first of these two properties that it is already a condition of the other; it can be demonstrated that it is a kind of configuration called a neutral solenoid.

The neutral solenoids have many characteristics in common with electrical strata in equilibrium over the surface of a conductor. In order to show this case better, we will use Green's formula, and we will write this in the following way:

$$\frac{1}{4\pi}\int\left(\varphi\frac{d\frac{1}{r}}{dn} - \frac{1}{r}\frac{d\varphi}{dn}\right)d\omega = \frac{1}{4\pi}\int\frac{\Delta^2\varphi}{r}\,dS + \epsilon\varphi_1.$$

Then we will observe that:

First, if ϕ be the magnetic potential of a system of masses and if ω is an equipotential closed surface that contains within itself all of the masses, we have over this surface the potential $\phi = \text{constant} = \phi_0$, and then

$$\frac{1}{4\pi}\int\frac{d\varphi}{dn}\frac{d\omega}{r} = (1 - \epsilon)\varphi_1 + \epsilon\varphi_0.$$

This formula contains all of the theory of the so-called level strata which, as is well known and as is derived from the previous equation, can be substituted for the masses at all external points ($\varepsilon = 0$); and for the internal points the action is zero where ($\varepsilon = 1$).

Second, if ϕ is the electromagnetic potential of a system of electric currents and ω is a closed surface orthogonal to the equipotential surfaces, which do not contain any currents inside of themselves, we have over these surfaces $d\phi/dn = 0$ and then

$$\frac{1}{4\pi}\int\varphi\frac{d\frac{1}{r}}{dn}d\omega = \epsilon\varphi_1.$$

This formula contains all of the theory of the so-called neutral solenoids which, as we have said before, and as we saw from the preceding equation, can be substituted for the electrical currents at all of the internal points ($\varepsilon = 1$), and the action is 0 for all of the external points ($\varepsilon = 0$).

The comparison between these two statements in which I have underlined the words and phrases that constitute the difference, illustrates the duality which exists for many problems, between electrostatics and electrodynamics, and shows reciprocity, to which I alluded before, because it is well known that the electricity in equilibrium over the surface of a conductor distributes itself in such a way as to form a level strata.[6]

Boltzmann has already spoken about this analogy.[7] He demonstrated that the potential that a neutral solenoid exerts upon itself is a minimum of all the configuration of systems of the currents which can circulate around the given surface ω subordinated to certain other conditions which it is not necessary to specify. From this, the result follows that in neutral solenoids the electrical currents are themselves kept in equilibrium, and this is the same as occurs in the distribution of static electricity upon the surface of a conductor. This minimal property is intimately connected with the so-called Dirichlet Principle because, in virtue of a formula that was demonstrated at the conclusion of the § 17 of the cited monograph (in which I extended Helmholtz's theorem in his paper on vortices), the potential of the neutral solenoid upon itself is equal to the product of 2π and the integral

$$\int\left[\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2+\left(\frac{\partial\varphi}{\partial z}\right)^2\right]dS$$

extended on all of the space S inside the solenoid.

A singular circumstance is that the general theorem demonstrated previously, which we can apply in a simple way for many problems, cannot be simply applied to the most simple solenoids, which are known as Ampère solenoids.[8] In such cases, the potential function of these solenoids, which are formed by elementary currents, both equal and equidistant, and which are perpendicular to the axis, can be immediately calculated from the potential of a single current, by integrating along the axis. It can be shown, however, that the method of calculating their potential is a special case of the calculation for the general solenoid Σ.

The function ϕ in such cases is not explicitly dependent upon x, y, z, but depends upon the equation $F(x, y, z, \phi) = 0$. In such a case, if we represent with Δ the sum of the squares of the first derivative with respect to x, y, and z we have

$$\Delta\varphi = \frac{\Delta F}{F'^2}, \quad \Delta^2\varphi = \frac{1}{F'}\left[\left(\frac{\Delta F}{F'}\right)' - \Delta^2 F\right],$$

where the prime indicates the partial derivative with respect to ϕ in F, and then also in ΔF. If we take as F a function which has the following forms

$$F = (x-\xi)\frac{d\xi}{d\varphi} + (y-\eta)\frac{d\eta}{d\varphi} + (z-\zeta)\frac{d\zeta}{d\varphi},$$

where ξ, η, ζ are the coordinates and ϕ is the arc of any line L, so that the surfaces (ϕ) are in this case planes normal to this line. In this hypothesis, we found that $\Delta F = 1$, and $\Delta^2 F = 0$ and then

$$\Delta\varphi = \frac{1}{F'^2}, \quad \Delta^2\varphi = -\frac{F''}{F'^3},$$

$$F' = (x-\xi)\xi'' + (y-\eta)\eta'' + (z-\zeta)\zeta'' - 1,$$

$$F'' = (x-\xi)\xi''' + (y-\eta)\eta''' + (z-\zeta)\zeta'''.$$

The x, y, z that are found in these equations by virtue of the equation $F = 0$, the coordinates of a point m existing in the plane normal to the line L at point μ, which has coordinates ξ, η, ζ and with parameter ϕ. In order to fix the position of m in such a plane, I imagine two orthogonal lines lying on the plane, determined by each point μ, well-defined for each value of ϕ, the normal principal and the perpendicular to the osculating plane I call u, v the coordinates of the point m with respect to these two axes. In such a way, the position of any point in space is determined by three values of the quantity ϕ, u, v, which three numbers are totally separate if the point remains in the region around the line L. Because of these conventions, and also because of the theorems that we know from differential geometry, calling γ and δ the curvature of the first and second species of the line L and at point ϕ, we have

$$F' = \gamma u - 1, \; F'' = \gamma' u - \gamma\delta v,$$

and from this we get $u = 0$, $v = 0$, if $\Delta\phi = 1$, and $\Delta^2\phi = 0$. This result can be formulated in the following manner: considering the arc ϕ as a function of the coordinates x, y, z of a point on a plane normal to a line L, at the variable extremity of the arc itself, the equations $\Delta\phi = 1$ and $\Delta^2\phi = 0$ are satisfied for the coordinates x, y, z, at all points on the line L.

In consequence of this property, the expressions $\Delta\phi - 1$, and $\Delta^2\phi$ are infinitely small in each point which is infinitely close to the line L, and are the same order of distance of this point from the line. Therefore if this line is the direction of an Ampère solenoid, in formula (2) we can pose

$$\Delta\phi, = 1, \; \Delta^2\phi = 0,$$

and thus,

$$P = 0$$

because, since derivative $d\phi/dn = 0$, on all the tubular surfaces, we have only an error of the third order, and the remaining quantities are of the second order. These quantities are the value of Π, which depend upon the two boundary sections, and because we have $d\phi/dn = \pm\sqrt{\Delta\phi} = \pm 1$ (the sign $+$ is relative to the origin of the arc, and the sign $-$ to the end), it is clear that the previous formula precisely reproduces the result which we know.

What remains now is the analytical demonstration of the equation (1), and we will first define the following lemma. Because of the condition of being single-valued which we ascribed to the function ϕ, we have, the transformations

$$\int \frac{\partial^2 \varphi}{\partial y \partial z} dS = - \int \frac{\partial \varphi}{\partial y} \frac{dz}{dn} d\omega = - \int \frac{\partial \varphi}{\partial z} \frac{dy}{dn} d\omega ,$$

then the relation follows

$$\int \left(\frac{\partial \varphi}{\partial y} \frac{dz}{dn} - \frac{\partial \varphi}{\partial z} \frac{dy}{dn} \right) d\omega = 0.$$

I will demonstrate that this is true also in the case of substituting the function ϕ/r for the functions ϕ. That is very clear when the point m_1 is external to the space S. But when m_1 is inside, the function ϕ/r becomes infinite in m_1, and we cannot use the previous transformation. I will suppose then, for a moment, that the surface ω be a spherical surface ω_1, whose center is m_1, with a finite radius r; then I shall rewrite the previous equation, substituting the product of $r(\phi/r)$ for ϕ. We have then, taking the derivative of the product

$$r \int \left(\frac{\partial \frac{\varphi}{r}}{\partial y} \frac{dz}{dn} - \frac{\partial \frac{\varphi}{r}}{\partial z} \frac{dy}{dn} \right) d\omega_1 +$$

$$\frac{1}{r} \int \varphi \left(\frac{\partial r}{\partial y} \frac{dz}{dn} - \frac{\partial r}{\partial z} \frac{dy}{dn} \right) d\omega = 0.$$

Now the element of the second integral is always 0 because of the hypothesis that we have

$$\frac{\partial r}{\partial x} = - \frac{dx}{dn}, \quad \frac{\partial r}{\partial y} = - \frac{dy}{dn}, \quad \frac{\partial r}{\partial z} = - \frac{dz}{dn},$$

then, in order to have $r > 0$, we have

$$\int \left(\frac{\partial \frac{\varphi}{r}}{\partial y} \frac{dz}{dn} - \frac{\partial \frac{\varphi}{r}}{\partial z} \frac{dy}{dn} \right) d\omega_1 = 0,$$

We have this equation that has the same form as the previous one, but refers to a spherical surface ω_1, with a finite radius with its center in m_1, where the function ϕ/r becomes infinite. We can see now if ω is any closed surface from which the point ω_1 is a finite distance, we can always inscribe a spherical surface ω which will not extend beyond the space S which is bounded by the surface ω. In the space between the two surfaces, the function ϕ/r satisfies the condition true for ϕ; then the integral

$$\int \left(\frac{\partial \frac{\varphi}{r}}{\partial y} \frac{dz}{dn} - \frac{\partial \frac{\varphi}{r}}{\partial z} \frac{dy}{dn} \right) d\omega$$

has the same value for the closed surface ω and thus for point m, at a finite distance from all of its points. (It would be equally provable that such an equation could exist, substituting for r a function of r, so that $r = 0$).

$$\int \left(\frac{\partial \frac{\varphi}{r}}{\partial y} \frac{dz}{dn} - \frac{\partial \frac{\varphi}{r}}{\partial z} \frac{dy}{dn} \right) d\omega = 0 \tag{3}$$

Returning now to the argument, I shall indicate with s, the arc of any of the lines (ϕ), and I shall determine the direction in the following manner: from a point (x, y, z) from this line (ϕ) I will produce two radii, one in the direction of the normal internal to ω, and the other in the direction of the minimal distance $d\,\sigma$ of the same point on the contiguous line ($\phi + d\phi$) which we take from the part where $d\phi$ is positive. Then I assume as the direction of the arcs s that is growing, that is, of the positive ds, the direction that is like the positive z axis, placed with respect to the positive x and y axis. In this hypothesis, through very easy geometrical steps we have

$$\begin{aligned} \frac{\partial \varphi}{\partial y} \frac{dz}{dn} - \frac{\partial \varphi}{\partial z} \frac{dy}{dn} &= - \frac{dx}{ds} \frac{d\varphi}{d\sigma}, \\ \frac{\partial \varphi}{\partial z} \frac{dx}{dn} - \frac{\partial \varphi}{\partial x} \frac{dz}{dn} &= - \frac{dy}{ds} \frac{d\varphi}{d\sigma}, \\ \frac{\partial \varphi}{\partial x} \frac{dy}{dn} - \frac{\partial \varphi}{\partial y} \frac{dx}{dn} &= - \frac{dz}{ds} \frac{d\varphi}{d\sigma}. \end{aligned} \tag{4}$$

Now it is well known that the components of magnetic action of a system of currents are given by

$$\begin{aligned} \frac{\partial \Phi}{\partial x_1} &= \frac{\partial Z}{\partial y_1} - \frac{\partial Y}{\partial z_1}, \\ \frac{\partial \Phi}{\partial y_1} &= \frac{\partial X}{\partial z_1} - \frac{\partial Z}{\partial x_1}, \\ \frac{\partial \Phi}{\partial z_1} &= \frac{\partial Y}{\partial x_1} - \frac{\partial X}{\partial y_1}, \end{aligned} \tag{5}$$

where

$$X = \int d\varphi \int \frac{dx}{r}, \quad Y = \int d\varphi \int \frac{dy}{r},$$

$$Z = \int d\varphi \int \frac{dz}{r}.$$

Then we have by virtue of equation (4), making $ds\, d\sigma = d\omega$,

$$X = -\int \left(\frac{\partial\varphi}{\partial y}\frac{dz}{dn} - \frac{\partial\varphi}{\partial z}\frac{dy}{dn} \right) \frac{d\omega}{r},$$

That means for equation (3)

$$X = \frac{\partial}{\partial z_1} \int \frac{dy}{dn}\frac{\varphi d\omega}{r} - \frac{\partial}{\partial y_1} \int \frac{dz}{dn}\frac{\varphi d\omega}{r},$$

and similarly

$$Y = \frac{\partial}{\partial x_1} \int \frac{dz}{dn}\frac{\varphi d\omega}{r} - \frac{\partial}{\partial z_1} \int \frac{dx}{dn}\frac{\varphi d\omega}{r},$$

$$Z = \frac{\partial}{\partial y_1} \int \frac{dx}{dn}\frac{\varphi d\omega}{r} - \frac{\partial}{\partial x_1} \int \frac{dy}{dn}\frac{\varphi d\omega}{r}.$$

Substituting these values in the second members of (5), and observing that we have

$$\Delta^2 \int \frac{dx}{dn}\frac{\varphi d\omega}{r} = 0,$$

$$\Delta^2 \int \frac{dy}{dn}\frac{\varphi d\omega}{r} = 0,$$

$$\Delta^2 \int \frac{dz}{dn}\frac{\varphi d\omega}{r} = 0,$$

$$\frac{\partial}{\partial x_1} \int \frac{dx}{dn}\frac{\varphi d\omega}{r} + \frac{\partial}{\partial y_1} \int \frac{dy}{dn}\frac{\varphi d\omega}{r} + \frac{\partial}{\partial z_1} \int \frac{dz}{dn}\frac{\varphi d\omega}{r}$$

$$= -\int \varphi \frac{d\frac{1}{r}}{dn} d\omega,$$

We find that the three derivatives of the potential Φ and of the functions

$$\int \varphi \frac{d\frac{1}{r}}{dn} d\omega \qquad (6)$$

are equal to each other. These two functions can be taken, one for the other, in the calculus of the components, and in this we establish equation (1), which is now demonstrated.

We observe that when the function ϕ is multivalued with one or more periodic forms, we can always, by making perpendicular cuts in the space S, (in a number equal to the forms) transform it into a single-valued function. The presence of such sections, each of which penetrates the space twice on either side of the cutting plane, on the new overall surface I'll call Ω, does not change the surface integrals whose elements contain ϕ as a factor. Then we need to write

$$\int \frac{dx}{dn}\frac{\varphi d\Omega}{r}, \quad \int \frac{dy}{dn}\frac{\varphi d\Omega}{r}, \quad \int \frac{dz}{dn}\frac{\varphi d\Omega}{r}$$

in place of

$$\int \frac{dx}{dn}\frac{\varphi d\omega}{r}, \quad \int \frac{dy}{dn}\frac{\varphi d\omega}{r}, \quad \int \frac{dz}{dn}\frac{\varphi d\omega}{r},$$

and then we have

$$\int \varphi \frac{d\frac{1}{r}}{dn} d\Omega$$

in place of

$$\int \varphi \frac{d\frac{1}{r}}{dn} d\omega,$$

that means of Φ. But this substitution does not influence the proposition that we made around equation (2), because the integrals P and Π are not changed by substituting the Ω [the surface after the cut] for ω.

This proposition then, is true, under the condition that the branches, in which the derivative of ϕ ceases to be single-valued, is finite and continuous, and external to the space S. And this is what we said above.

In other words, the integral

$$\int \varphi \frac{d\frac{1}{r}}{dn} d\Omega$$

is the same as the primitive

$$\int \varphi \frac{d\frac{1}{r}}{dn} d\omega$$

added to the sum of the potential of the currents which cirulate around the transverse sections, with equal intensity with respect to the periodic form and this is explained in § 70 of my monograph.

Notes

1. *Nachrichten von der K. Gesellschaft d. W.*, Göttingen, (1870); *Annalen der Physik und Chemie von Poggendorff*, Vol. CXLV, (1872).
2. Memoria XXXV of this work (*Opere*), Vol. II
3. *Journal für die reine und angewandte Mathematik*, Vol. LXIX (1868), pp. 125-126.
4. *Journal für die reine und angewandte Mathematik*, Vol. XXXVII, (1848), p. 47.
5. *Sitzungberichte der K. Böhmischen Gesellschaft d. W.*, (1871), p. 25.
6. *Cf.* Section XI of the excellent *Teorica delle forze che agiscono secondo la legge di Newton*, (Theory of the forces which agitate according to Newton's law) published by Betti in this same journal.
7. *Journal für die reine und angewandte Mathematik*, Vol. LXIII (1871), pp. 116-119.
8. *Mémoires de l'Académie Royale des Sciences de l'Institut de France*, Vol. VI (1823), p. 175.