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Abstract

A key obstruction to further developments in nonlinear quantum theory is the lack of obvious procedures for their quantization. How should we construct candidate nonlinear theories when the orthodox quantization techniques can produce only linear partial differential equations? To escape this impasse, we return to an early idea of Schrödinger and seek quantized values via the *nonlinear* generalization of linear spectral theory, realized via self–energy terms.

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I. INTODUCTION

The utility of the superposition principle is well–established, but we may well ask if Nature accords it the full force of law[1]. More interestingly, we may ask what *larger* physical theories are possible if one permits small violations of it. Perhaps if the superposition principle were an approximate law, a different interpretation is made possible. Such an alternative interpretation might permit novel extensions in the scope of quantum field theory — to include gravitation alongside the other interactions. That is the larger aim behind our studies in nonlinear quantum theory.

Here we explore adding nonlinear terms as a means to rescue the Schrödinger interpretation from oblivion[2]. First, we consider how nonlinear terms can widen the possible class of reduction mechanisms in theories of quantum measurement based on decoherence. Second, we show how nonlinear equations are compatible with the Schrödinger interpretation. Finally, we add a nonlinear gravitational self–energy term to suppress macroscopic dispersion. The result is a quantized nonlinear theory that overcomes the key failings of the original Schrödinger viewpoint[3, 4].

II. PHYSICAL INTERPRETATION

Interpretation plays a dual role in physics. It ties the mathematical constructs of a theory with elements of the physical world, and offers a framework to conceptualise the physical world when formulating new theories. Less obvious is the role physical interpretation plays in fixing the intended scope of a theory. Figure 1 elaborates this point, comparing three approaches to the interpretation of quantum mechanics.

Figure 1a depicts the Copenhagen interpretation. This divides the universe into two: an *observer* (an eyeball) and the *observed* (a superposed wavepacket). There is no attempt to explain measurement, only a rule for extracting the probabilitities associated with each branch in the superposition of possible measuremnt outcomes.

Figure 1b depicts a popular interpetation based on environmental decoherence. Coherent superpositions are extremely fragile in macrosystems with huge numbers of constituents. The off–diagonal phase relations in the density matrix are rapidly destroyed when a coherent quantum system (the observed) interacts with a laboratory instrument (the observer). Hence one can model the mechanism for measurement, but we cannot say where in the chain a measurement takes place.

Figure 1c depicts orthodox decoherence augmented by an additional physical mechanism which acts to reduce the remnant diagonal elements in the density matrix. Nonlinear theories permit interactions of this kind[3, 4]. One possible example is a gravitational force of localization which acts to condense any putative macroscopic superposition into just one manifest branch. The Schrödinger interpretation could be readily applied if we identify the reduced density matrix with the laboratory reality.

III. SCHRÖDINGER NONLINEAR THEORIES

Consider a nonlinear wave equation of the form[3, 4]

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x},t) = -\frac{\hbar^2}{2m}\nabla^2_{\mathbf{x}}\psi(\mathbf{x},t) + V[\psi,\psi^*]\psi(\mathbf{x},t), \tag{1}$$

where all nonlinearity resides in the potential $V[\psi,\psi^*]$, such as $V[\psi,\psi^*] = \pm\kappa|\psi(\mathbf{x},t)|^2$. While incompatible with the Copenhagen interpretation, such equations support an interpretation based on the ideas of Schrödinger.

Firstly, one can generalize the existing linear spectral theory and define nonlinear *eigenstates* by the condition

$$\nabla^2_{\mathbf{x}}\psi(\mathbf{x},t) + \frac{2m}{\hbar^2}(E - V[\psi,\psi^*])\psi(\mathbf{x},t) = 0, \tag{2}$$

where the *eigenvalue* E is numerically equal to the energy functional

$$H[\psi,\psi^*] = \int_{\mathbf{R}^3} d^3\mathbf{x}\psi(\mathbf{x},t)\left\{-\frac{\hbar^2}{2m}\nabla^2_{\mathbf{x}}\psi(\mathbf{x},t) + V[\psi,\psi^*]\right\}\psi^\star(\mathbf{x},t), \tag{3}$$

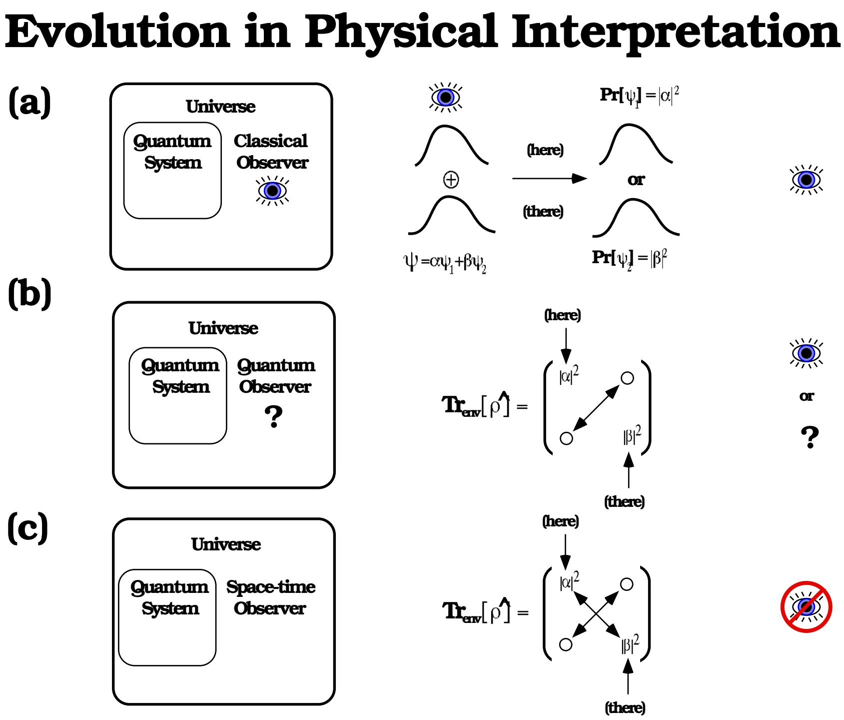

FIG. 1: Three levels of sophistication in the physical interpretation of a quantum measurement of position for a wavepacket found “here” with probability $|\alpha|^2$, or “there” with probability $|\beta|^2$. One can: (a) submerge all mystery in an “observer”; (b) invoke decoherence to wash out off–diagonal phase relations; or (c) posit a further level of decoherence, acting along the diagonal, to crystallize individual events.

and exhibits a spectral sequence of quantized values. Secondly, the quantity

$$j(\mathbf{x},t) = \frac{\hbar}{2mi}\left\{\psi^*(\mathbf{x},t)\nabla_{\mathbf{x}}\psi(\mathbf{x},t) - \psi(\mathbf{x},t)\nabla_{\mathbf{x}}\psi^*(\mathbf{x},t)\right\} \tag{4}$$

is a conserved current provided only that $V[\psi,\psi^*]$ is real–valued. Finally, whereas the superposition principle is violated, and we cannot in general expand each state as a linear superposition of orthogonal eigenstates, we *can* write a linear equation for the evolution of a *density matrix functional* $\rho[\psi,\psi^*;t]$. In particular, we could try

$$i\hbar\frac{\partial\rho}{\partial t} = \{\rho, H\}_{\mathrm{FPB}} \tag{5}$$

where the Functional Poisson Bracket (FPB) is defined

$$\{F,G\}_{\mathrm{FPB}} \equiv \int_{\mathbf{R}^3} d^3\mathbf{x}\left\{\frac{\delta F}{\delta\psi(\mathbf{x},t)}\frac{\delta G}{\delta\psi^*(\mathbf{x},t)} - \frac{\delta F}{\delta\psi^*(\mathbf{x},t)}\frac{\delta G}{\delta\psi(\mathbf{x},t)}\right\} \tag{6}$$

and we consider *impure* density matrices to be defined as suitable positive weighted mixtures of *pure states*. This is most easily done using the definition

$$\hat{\rho}(\mathbf{x}, \mathbf{x}'; t) \equiv \int d\psi d\psi^* \psi(\mathbf{x}) \psi^*(\mathbf{x}') \rho[\psi, \psi^*; t] \tag{7}$$

where $\rho[\psi, \psi^*; t]$ is an arbitrary *non–negative* functional on the pure states ψ, and the integration is performed with respect to the canonical invariant measure $d\psi d\psi^*$. Then a pure density matrix functional has a measure $\rho[\psi, \psi^*; t]$ concentrated on a single time–dependent pure state $\psi(t)$. It follows that the eigenstates defined above are also stationary solutions of this *generalized Liouville equation*. Hence they comprise the possible steady states for a closed nonlinear quantum system. Finally, since $V[\psi, \psi^*]$ depends on ψ, we see that cross–coupling of "diagonal entries", i.e. $\rho[\psi, \psi^*](\mathbf{x}, \mathbf{x}; t)$, is possible, as in Figure 1c. Thus one can try and extend the decoherence scenario, including a density matrix framework, to nonlinear theories.

IV. SCHRÖDINGER QUANTUM GRAVITY

The preceding indicates a plausible way for interpreting nonlinear equations, but offers no concrete physical origin for nonlinearity. The existing quantization schemes generate only linear equations, so we must identify some physically consistent origin for a nonlinear quantization, i.e. nonlinearity must arise from a physical interaction.

Further, the Schrödinger interpretation suffers from a well–known difficulty with dispersion. Any nonlinear term we propose must suppress macroscopic dispersion, but leave the microscopic dispersion untouched.

To meet both demands we consider a model where *all* physical nonlinearities are traced to *self–interactions*, and consider invoking the gravitational self–energy as a generic localizing mechanism to suppress macroscopic dispersion[3].

Interestingly, the Schrödinger interpretation permits a new approach to quantizing self–interactions. Rather than attempt to second quantize the gravitational field and incorporate self–energy as the 1–loop, and higher, perturbative corrections, we can now attempt a complete non–perturbative treatment from the outset.

Specifically, we follow Schrödinger and re–interpret the square modulus of the wavefunction as a mass–density, i.e. we consider

$$\rho(\mathbf{x},t) = m\psi(\mathbf{x},t)\psi^*(\mathbf{x},t) \tag{8}$$

as the mass density for our particle. The correspondence principle dictates that the self–field behaves as $1/r$ for larger r. Hence we adopt the Poisson equation

$$\nabla^2 V_{\rm grav}(\mathbf{x}) = 4\pi G m \rho(\mathbf{x},t), \tag{9}$$

as its source, and obtain the *gravitational self–potential*

$$V_{\rm grav}(\mathbf{x}) = -Gm \int \frac{\psi^*(\tilde{\mathbf{x}},t)\psi(\tilde{\mathbf{x}},t)}{|\mathbf{x}-\tilde{\mathbf{x}}|}\, d^3\tilde{\mathbf{x}}. \tag{10}$$

Coupling (10) back upon the particle we find

$$E_{\rm grav} = -\frac{Gm^2}{2}\int\int \frac{\psi^*(\tilde{\mathbf{x}},t)\psi^*(\mathbf{x},t)\psi(\tilde{\mathbf{x}},t)\psi(\mathbf{x},t)}{|\mathbf{x}-\tilde{\mathbf{x}}|}\, d^3\tilde{\mathbf{x}} d^3\mathbf{x}, \tag{11}$$

while the wave–equation becomes

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x},t) = \left\{-\frac{\hbar^2}{2m}\nabla^2 - \frac{Gm^2}{N[\psi,\psi^*]}\int \frac{\psi^*(\tilde{\mathbf{x}},t)\psi(\tilde{\mathbf{x}},t)}{|\mathbf{x}-\tilde{\mathbf{x}}|}\, d^3\tilde{\mathbf{x}} - \frac{E_{\rm grav}}{N[\psi,\psi^*]}\right\}\psi(\mathbf{x},t), \tag{12}$$

where the factor

$$N[\psi,\psi^*] \equiv \int \psi^*(\mathbf{x},t)\psi(\mathbf{x},t)\, d^3\mathbf{x}, \tag{13}$$

ensures nomralization, and the term $E_{\rm grav}/N[\psi,\psi^*]$, subtracted in (12), is there to avoid double–counting in the calculation of the total energy functional[3].

V. QUANTIZATION AS A NONLINEAR EIGENVALUE PROBLEM

In Figure 2 we display the first two eigenstates of the equation (12). These form a spectral series like that for the hydrogen atom[4]. The nonlinear spectrum can thus be interpreted as offering a *quantization* of the original classical problem. Hence nonlinear quantization appears as a natural result of incorporating self–interactions.

VI. CONCLUSION

My aim was to tie the Schrödinger interpretation with a nonlinear quantization founded in self–interactions. This approach seems capable of considerable further development. For

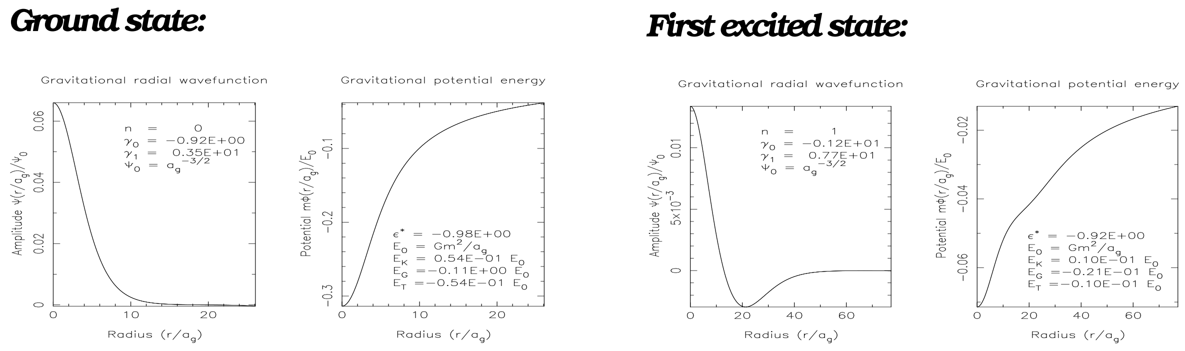

FIG. 2: The quantized ground state, first excited state, and their self–potentials.

now, we point out that earlier work by Barut[5], on a theory of *self–field electrodynamics*, appears compatible with the scheme of interpretation adopted here. Thus it seems reasonable to pursue a combined theory of gravity and electromagnetism as a further test of the consistency of this viewpoint.

Acknowledgments

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